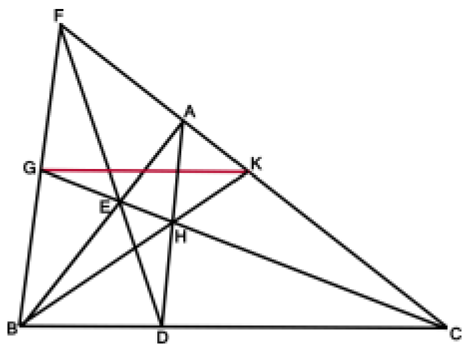
Cash Award question of July 2023



In ΔABC, D is a point on BC such that BD: DC = 1:2. E is a point on AB. DE & CA produced meet at F. CE meets AD at H and CE produced meets FB at G. BH produced meets FC at K. Prove: GK ∥ BC.

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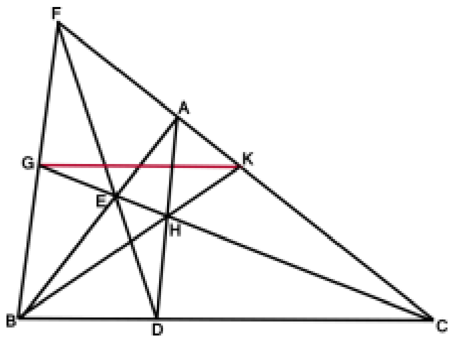


Author's Solution

The author is giving two solutions to this problem.

Solution : 1

In ΔABC, CE, BK & AD are cevians concurrent at H.

$\therefore$ As per Ceva's theorem, $\frac{BD}{DC} \times \frac{CK}{KA} \times \frac{AE}{EB} = 1$ ----------------(1)

For $\Delta ABC, DEF$ is a transversal,

As per Menelaus theorem, $\frac{BD}{DC} \times \frac{CF}{FA} \times \frac{AE}{EB} = 1$ ----------------------- (2)

(1) & (2) $\rightarrow \frac{CK}{KA} = \frac{CF}{FA} = m$ (say) -------------------- (3)

$\mathrm{m} = \frac{\mathrm{CK+CF}}{\mathrm{KA+FA}} = \frac{\mathrm{2CK+KF}}{\mathrm{KF}}$ [Componendo-Dividendo Rule]

$m - 1 = \frac{2CK + KF - KF}{KF} = \frac{2CK}{KF}$

$\Longrightarrow \frac{CK}{KF} = \frac{(m-1)}{2}$ --------------------- (4)

(3) $\longrightarrow \frac{CF}{FA} = m$

$\therefore \frac{CF}{FA} - 1 = m - 1$

$\frac{CA}{AF} = (m - 1)$ --------------------- (5)

In $\Delta FBC, FD, CG$ & BA are cevians concurrent at E.

$\therefore \frac{BD}{DC} \times \frac{CA}{AF} \times \frac{FG}{GB} = 1$ ------------------------------ (6)

(5) & (6) $\longrightarrow$

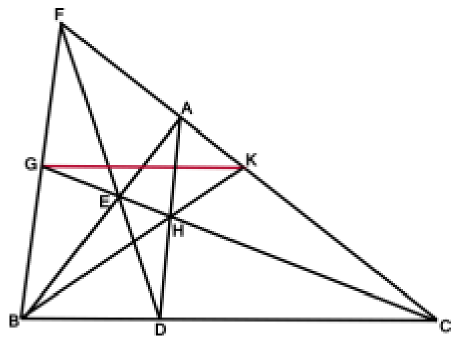
$\frac{1}{2} \times (m - 1) \times \frac{FG}{GB} = 1$ $[\because \frac{BD}{DC} = \frac{1}{2}\ given]$

$\Rightarrow \frac{GB}{GF} = \frac{(m-1)}{2}$ -----------------------*(7)*

(4) & (7) $\longrightarrow \frac{CK}{KF} = \frac{BG}{GF}$

$\therefore GK \parallel BC$ -------------------------Proved.

Solution : 2



Let CK=mAK

$\frac{CK}{AK} = m$ --------------------- (1)

In $\Delta ABC, AD, BK$ & CE are cevians concurrent at H.

$\frac{CK}{AK} \times \frac{AE}{EB} \times \frac{BD}{DC} = 1$

$m \times \frac{AE}{EB} \times \frac{1}{2} = 1$ $\quad (\frac{BD}{DC} = \frac{1}{2} given)$

$\Rightarrow \frac{AE}{EB} = \frac{2}{m}$ ————————————(2)

For $\Delta\ ABK$, EHC is a transversal.

$\therefore \frac{AE}{EB} \times \frac{BH}{HK} \times \frac{KC}{CA} = 1$

$\frac{2}{m} \times \frac{BH}{HK} \times \frac{m}{m+1} = 1$

$\frac{BH}{HK} = \frac{m+1}{2}$ ———————————— (3)

For ΔAEC, KHB is a transversal.

$\therefore \frac{CK}{KA} \times \frac{AB}{BE} \times \frac{EH}{HC} = 1$ ------------------------- (4)

(2) → $\frac{AB}{BE} = \frac{m+2}{m}$ ————————————(5)

(4) & (5) →

$\frac{m \times (m+2)}{m} \times \frac{EH}{HC} = 1$

$\frac{EH}{HC} = \frac{1}{(m+2)}$ --------------------- (6)

For ΔFBC, FD, BA & CG are cevians concurrent at E and AD cuts CG at H. Therefore, as per concurrency theorem, (the statement and proof of the concurrency theorem is available in this website vide page no-9 of the book "The Geometry of Concurrency", put up in this site),

$\frac{EH}{HC} = \frac{EG}{GC}$

$\frac{EH+EG}{HC+GC}$ (Componendo-Dividendo Rule) --------------------- (7)

(6) & (7) $\rightarrow \frac{1}{m+2} = \frac{EH+EG}{HC+GC} = \frac{GH}{2HC+GH}$

$\Rightarrow \frac{2HC+GH}{GH} = (m+2)$

$\frac{2HC+GH}{GH} - 1 = m + 2 - 1$

$\frac{2HC}{GH} = m + 1$

$\frac{HC}{GH} = \frac{m+1}{2}$ ________________(8)

(3) & (8) $\longrightarrow$

$\frac{BH}{HK} = \frac{CH}{HG}$

$\therefore GK \parallel BC$----------------Proved

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